The five-loop beta function of Yang-Mills theory with fermions

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based on JHEP 02 (2017) 090, arXiv:1701.01404 [hep-ph]

2025 ICBS

Outline

Introduction

Strong interaction

Fundamental force binding quarks into nucleons

Deep understanding essential for analyses of experiments at the LHC

Proton collisions
→ many strongly interacting particles

High-precision experimental data
↔ High-precision theoretical predictions

Precision is key

Figure from arXiv:2410.22148 (Sherpa 3)

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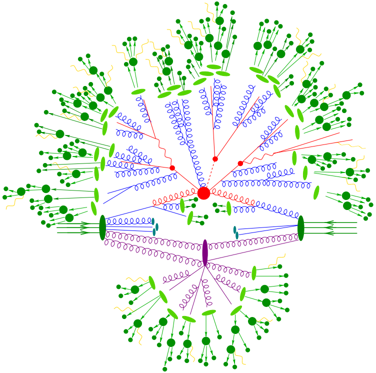

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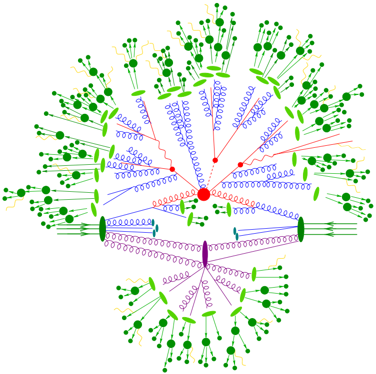

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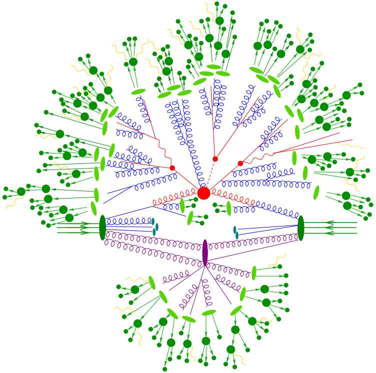

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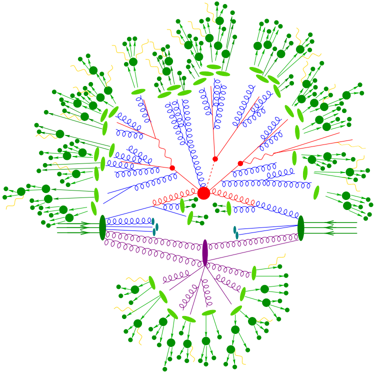

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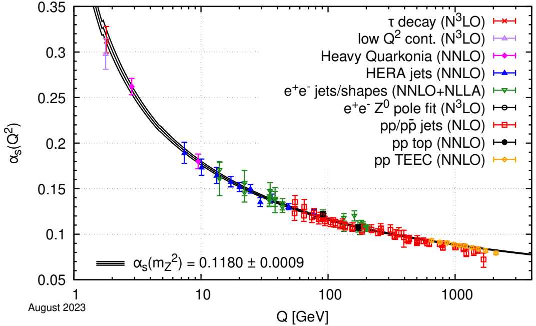
Running coupling constant

Strength varies with the energy scale



Huston, Rabbertz, Zanderighi (PDG '23)

Weaker at large scales
(asymptotic freedom)

$$\frac{d}{d\ln Q^2}\left(\frac{\alpha_s}{4\pi}\right) = \beta(\alpha_s) = -\beta_0\left(\frac{\alpha_s}{4\pi}\right)^2 - \beta_1\left(\frac{\alpha_s}{4\pi}\right)^3 - \beta_2\left(\frac{\alpha_s}{4\pi}\right)^4 - \beta_3\left(\frac{\alpha_s}{4\pi}\right)^5 - \beta_4\left(\frac{\alpha_s}{4\pi}\right)^6 - \cdots$$

LO (1-loop) NLO (2-loop) … N^{4}LO (5-loop)

Running coupling constant

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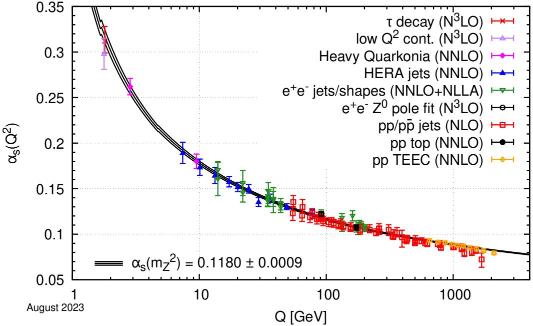

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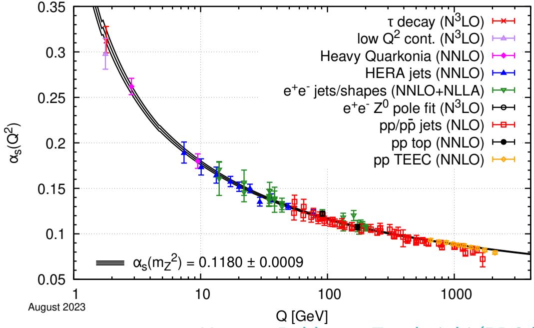

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LO (1-loop)   NLO (2-loop)   …   N^{4}LO (5-loop)

Results

Group notation (1)

Fermions transform in a representation of a simple compact Lie group
(matter particles)

$$[T^a, T^b] = i f^{abc} T^c$$

- T^a are generators of the representation of the fermions

  trace normalisation: $\mathrm{Tr}(T^a T^b) = T_F \delta^{ab}$

- $(C^a)_{bc} = -i f^{abc}$ are generators of the adjoint representation
  (gauge group; force-carrying particles)

Quadratic Casimir invariants C_F and C_A:

$$(T^a T^a)_{ij} = C_F \delta_{ij}, \qquad f^{acd} f^{bcd} = C_A \delta^{ab}$$

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Group notation (2)

At $L \geq 4$ loops, we also need quartic group invariants

$$\frac{d_A^{abcd} d_A^{abcd}}{N_A}, \quad \frac{d_F^{abcd} d_A^{abcd}}{N_A}, \quad \frac{d_F^{abcd} d_F^{abcd}}{N_A}$$

$$d_F^{abcd} = \frac{1}{6} \mathrm{Tr}\left(T^a T^b T^c T^d + \text{permutations} \right)$$

$$d_A^{abcd} = \frac{1}{6} \mathrm{Tr}\left(C^a C^b C^c C^d + \text{permutations} \right)$$

* No symmetric tensors with an odd number of indices (Furry's theorem)

- T^a are generators of the fermionic representation
- C^a are generators of the adjoint representation
- N_A is the dimension of the adjoint representation

$$\beta(\alpha_s) = -\sum_{i=0}^{\infty} \beta_i \left(\frac{\alpha_s}{4\pi}\right)^{i+2}$$

Old results

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f$$

Vanyashin, Terent'ev '65; Khriplovich '69; 't Hooft '72; Gross, Wilczek '73; Politzer '73

$$\beta_1 = \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_F n_f - 4 C_F T_F n_f$$

Caswell '74; Jones '74; Egorian, Tarasov '79

$$\beta_2 = \frac{2857}{54} C_A^3 - \frac{1415}{27} C_A^2 T_F n_f - \frac{205}{9} C_F C_A T_F n_f + 2 C_F^2 T_F n_f + \frac{44}{9} C_F T_F^2 n_f^2 + \frac{158}{27} C_A T_F^2 n_f^2$$

Tarasov, Vladimirov, Zharkov '80; Larin, Vermaseren '93

$$\begin{aligned}\beta_3 = {} & C_A^4 \left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right) + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right) + C_A^3 T_F n_f \left(-\frac{39143}{81} + \frac{136}{3}\zeta_3\right) \\ & + C_A^2 C_F T_F n_f \left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right) + C_A C_F^2 T_F n_f \left(-\frac{4204}{27} + \frac{352}{9}\zeta_3\right) + \frac{d_F^{abcd} d_A^{abcd}}{N_A} n_f \left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right) \\ & + 46 C_F^3 T_F n_f + C_A^2 T_F^2 n_f^2 \left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right) + C_F^2 T_F^2 n_f^2 \left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right) + C_A C_F T_F^2 n_f^2 \left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right) \\ & + \frac{d_F^{abcd} d_F^{abcd}}{N_A} n_f^2 \left(-\frac{704}{9} + \frac{512}{3}\zeta_3\right) + \frac{424}{243} C_A T_F^3 n_f^3 + \frac{1232}{243} C_F T_F^3 n_f^3\end{aligned}$$

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* n_f is the number of fermions
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$$\beta(\alpha_s) = -\sum_{i=0}^{\infty} \beta_i \left(\frac{\alpha_s}{4\pi}\right)^{i+2}$$

Result

$$\begin{aligned}
\beta_4 = {} & C_A^5 \left(\frac{8296235}{3888} - \frac{1630}{81}\zeta_3 + \frac{121}{6}\zeta_4 - \frac{1045}{9}\zeta_5\right) + \frac{d_A^{abcd} d_A^{abcd}}{N_A} C_A \left(-\frac{514}{3} + \frac{18716}{3}\zeta_3 - 968\zeta_4 - \frac{15400}{3}\zeta_5\right) \\
& + C_A^4 T_F n_f \left(-\frac{5048959}{972} + \frac{10505}{81}\zeta_3 - \frac{583}{3}\zeta_4 + 1230\zeta_5\right) + C_A^3 C_F T_F n_f \left(\frac{8141995}{1944} + 146\zeta_3 + \frac{902}{3}\zeta_4 - \frac{8720}{3}\zeta_5\right) \\
& + C_A^2 C_F^2 T_F n_f \left(-\frac{548732}{81} - \frac{50581}{27}\zeta_3 - \frac{484}{3}\zeta_4 + \frac{12820}{3}\zeta_5\right) + C_A C_F^3 T_F n_f \left(3717 + \frac{5696}{3}\zeta_3 - \frac{7480}{3}\zeta_5\right) - C_F^4 T_F n_f \left(\frac{4157}{6} + 128\zeta_3\right) \\
& + \frac{d_A^{abcd} d_A^{abcd}}{N_A} T_F n_f \left(\frac{904}{9} - \frac{20752}{9}\zeta_3 + 352\zeta_4 + \frac{4000}{9}\zeta_5\right) + \frac{d_F^{abcd} d_A^{abcd}}{N_A} C_A n_f \left(\frac{11312}{9} - \frac{127736}{9}\zeta_3 + 2288\zeta_4 + \frac{67520}{9}\zeta_5\right) \\
& + \frac{d_F^{abcd} d_A^{abcd}}{N_A} C_F n_f \left(-320 + \frac{1280}{3}\zeta_3 + \frac{6400}{3}\zeta_5\right) + C_A^3 T_F^2 n_f^2 \left(\frac{843067}{486} + \frac{18446}{27}\zeta_3 - \frac{104}{3}\zeta_4 - \frac{2200}{3}\zeta_5\right) \\
& + C_A^2 C_F T_F^2 n_f^2 \left(\frac{5701}{162} + \frac{26452}{27}\zeta_3 - \frac{944}{3}\zeta_4 + \frac{1600}{3}\zeta_5\right) + C_F^2 C_A T_F^2 n_f^2 \left(\frac{31583}{18} - \frac{28628}{27}\zeta_3 + \frac{1144}{3}\zeta_4 - \frac{4400}{3}\zeta_5\right) \\
& + C_F^3 T_F^2 n_f^2 \left(-\frac{5018}{9} - \frac{2144}{3}\zeta_3 + \frac{4640}{3}\zeta_5\right) + \frac{d_F^{abcd} d_A^{abcd}}{N_A} T_F n_f^2 \left(-\frac{3680}{9} + \frac{40160}{9}\zeta_3 - 832\zeta_4 - \frac{1280}{9}\zeta_5\right) \\
& + \frac{d_F^{abcd} d_F^{abcd}}{N_A} C_A n_f^2 \left(-\frac{7184}{3} + \frac{40336}{9}\zeta_3 - 704\zeta_4 + \frac{2240}{9}\zeta_5\right) + \frac{d_F^{abcd} d_F^{abcd}}{N_A} C_F n_f^2 \left(\frac{4160}{3} + \frac{5120}{3}\zeta_3 - \frac{12800}{3}\zeta_5\right) \\
& + C_A^2 T_F^3 n_f^3 \left(-\frac{2077}{27} - \frac{9736}{81}\zeta_3 + \frac{112}{3}\zeta_4 + \frac{320}{9}\zeta_5\right) + C_A C_F T_F^3 n_f^3 \left(-\frac{736}{81} - \frac{5680}{27}\zeta_3 + \frac{224}{3}\zeta_4\right) + C_F^2 T_F^3 n_f^3 \left(-\frac{9922}{81} + \frac{7616}{27}\zeta_3 - \frac{352}{3}\zeta_4\right) \\
& + \frac{d_F^{abcd} d_F^{abcd}}{N_A} T_F n_f^3 \left(\frac{3520}{9} - \frac{2624}{3}\zeta_3 + 256\zeta_4 + \frac{1280}{3}\zeta_5\right) + C_A T_F^4 n_f^4 \left(\frac{916}{243} - \frac{640}{81}\zeta_3\right) - C_F T_F^4 n_f^4 \left(\frac{856}{243} + \frac{128}{27}\zeta_3\right)
\end{aligned}$$

Timeline of β_4

$$\beta_4 = c_0 + c_1 n_f + c_2 n_f^2 + c_3 n_f^3 + c_4 n_f^4$$

- 1996 Gracey (large-n_f limit; n_f^4-term only)
- 2012 Baikov, Chetyrkin, Kühn, Rittinger (QED)
- 2016 Baikov, Chetyrkin, Kühn (SU(3))
- 2016 Luthe, Maier, Marquard, Schröder (generic group but n_f^4 and n_f^3-terms only)
- 2017 Herzog, Ruijl, TU, Vermaseren, Vogt (generic group)
- 2017 Luthe, Maier, Marquard, Schröder (generic group) Confirmed our results

1995 2000 2005 2010 2015 2020

Other QCD renormalisation constants at 5-loop in $\overline{\text{MS}}$:

2014 Baikov, Chetyrkin, Kühn, quark mass and field anomalous dimensions (SU(3))

2016 Luthe, Maier, Marquard, Schröder, quark mass and field anomalous dimensions

2017 Luthe, Maier, Marquard, Schröder, ghost field and ghost-gluon vertex anomalous dimensions

2017 Baikov, Chetyrkin, Kühn, quark mass dimensions

2017 Chetyrkin, Falcioni, Herzog, Vermaseren, complete dependence on the covariant gauge parameter ξ

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1995 2000 2005 2010 2015 2020

Baikov, Chetyrkin, Kühn: computed β_4 for SU(3), QCD describing the real world strong interaction

Our work: computed β_4 not only for SU(3) but for a generic group
first complete validation by a second independent calculation,
applicable for other QCD-like theories

Timeline of β_4

$$\beta_4 = c_0 + c_1 n_f + c_2 n_f^2 + c_3 n_f^3 + c_4 n_f^4$$

- 1996 Gracey (large-n_f limit; n_f^4-term only)
- 2012 Baikov, Chetyrkin, Kühn, Rittinger (QED)
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Strategy

Perturbative QFT: crash cource (I)

Start from the Yang-Mills Lagrangian with fermions (∋ QCD):

$$\mathscr{L}_{\text{cl}} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \sum_f \bar{\psi}_f (i\not{D} - m_f)\psi_f$$

gauge bosons   fermions

If the strong coupling α_s is relatively small, amplitudes can be computed by perturbative expansion. Draw all possible Feynman diagrams using:

freely propagating fermions and gauge bosons

interaction vertices

Diagrams drawn with TikZ-FeynHand

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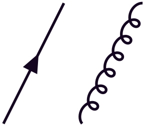

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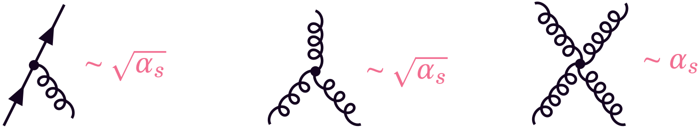
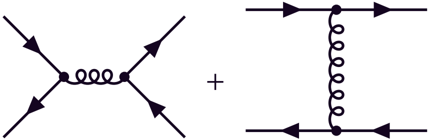


interaction vertices

Perturbative QFT: crash cource (II)

You can draw diagrams for a reaction amplitude, e.g., for $q\bar{q} \to q\bar{q}$:

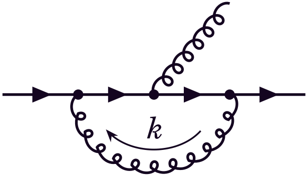


$\sim \alpha_s$

Each diagram can be translated into a mathematical expression via Feynman rules

Perturbative QFT: crash cource (III)

If a loop appear in your diagram, quantum mechanics tells you that all possible momentum states of the internal line must be summed up, leading to an integral (Feynman integral), often divergent



$$\sim \int \frac{d^4k}{k^4}$$ (logarithmically divergent)

Ultraviolet (UV) divergence: $k \to \infty$

Infrared (IR) divergence: $k \to 0$ (or collinear)

Perturbative QFT: crash cource (IV)

Dimensional regularisation: change the dimension of spacetime from 4 to $D = 4 - 2\epsilon$

Bollini, Giambiagi '72; 't Hooft, Veltman '72

Divergences are expressed as poles $\frac{1}{\epsilon^n}$

Good news:

- IR divergences are canceled out in the end (KLM theorem)
- UV divergences can be removed by renormalisation
  Scale evolution (β func.) is related to UV divergences, i.e., renormalisation constants $\Rightarrow$ We only need UV divergent part

Bad news:

- Multi-loop Feynman integrals are notoriously difficult to evaluate

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Abbott '81; Abbott, Grisaru Schaefer '83

Split the gauge field into the classical background field and additional quantum fluctuations

The coupling renormalisation constant can be obtained from the 2-point function of the background field

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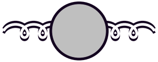
Only need to compute the divergence of 

Infrared rearrangement (IRR)

Superficial (i.e., overall) UV divergence: coming from the region where all loop momenta go to ∞

All mass scales (internal masses and external momenta) can be ignored

Vladimirov '80

Log-divergences are mass-independent in the dimensional regularisation

Rearrange diagram in such a way that the evaluation becomes easier

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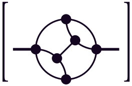
sup. UV div.  = sup. UV div.

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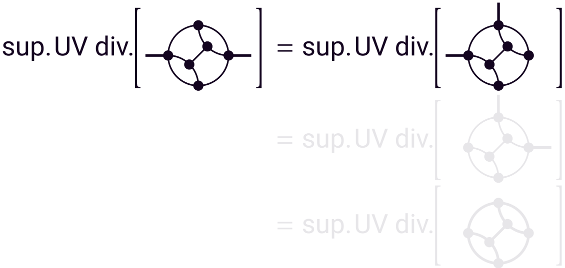

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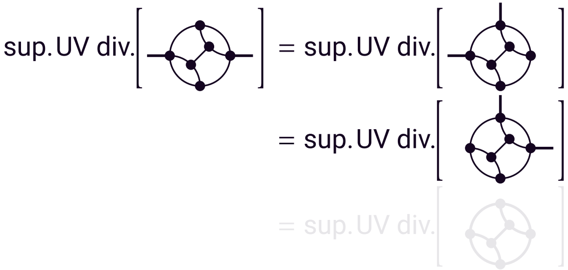

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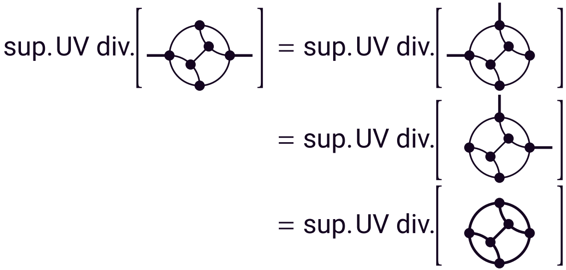

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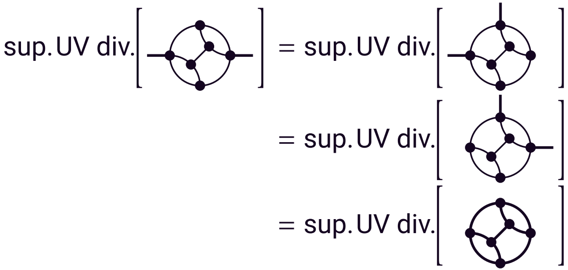

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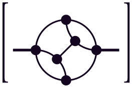
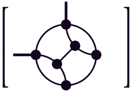
sup. UV div.  = sup. UV div. 

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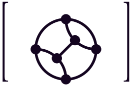
= sup. UV div. 

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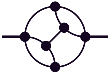
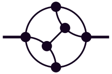
Computing pole parts

div.[] = sup. UV div.[] + (UV/IR subdivergences)

L-loops $\qquad$ L-loops $\qquad$ lower loops

R*

= sup. UV div.[ ] integrate 1-loop

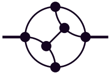
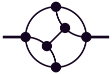
= sup. UV div.[ ]

(local) R*-operation:
generalisation of the BPHZ R-operation,
diagrammatically identifying/subtracting both UV and IR

Chetyrkin, Tkachov '82; Chetyrkin, Smirnov '83, '84;
Extension for arbitrary numerator structure: Herzog, Ruijl '17;
Hopf algebra structure: Beekveldt, Borinsky, Herzog '20

Poles of 5-loop diagrams can be computed as 4-loop massless propagator diagrams

Computing pole parts

div. [] = sup. UV div. [] + (UV/IR subdivergences)

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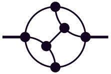
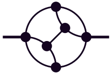
div. [] = sup. UV div. [] + (UV/IR subdivergences)

L-loops — effectively $(L-1)$-loops — lower loops

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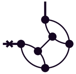
= sup. UV div. []

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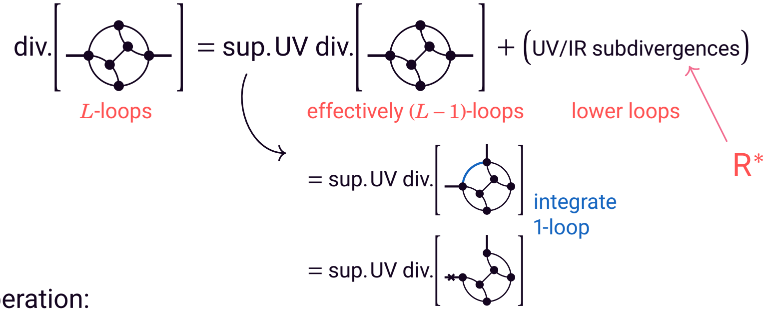

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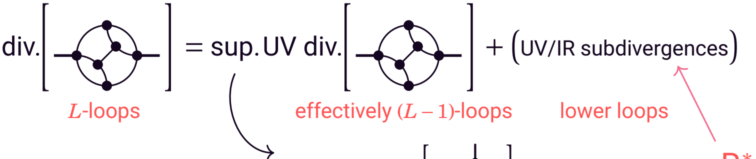

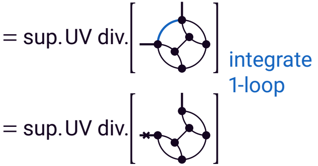

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Forcer

evaluating 4-loop massless propagator-type integrals

Forcer program

Program implemented in the FORM language (→ next page)
for 4-loop massless propagator-type scalar Feynman integrals

Ruijl, TU, Vermaseren '17

`https://github.com/benruijl/forcer`

Extends the 3-loop Mincer approach to 4-loops

Chetyrkin, Tkachov '81; Schoonschip version: Gorishny, Larin, Surguladze, Tkachov '89;
FORM version: Larin, Tkachov, Vermaseren '91

Heavily used for, e.g., 4-loop QCD calculations

Davies, Vogt, Ruijl, TU, Vermaseren '16; Ruijl, TU, Vermaseren, Vogt '17; Moch, Ruijl, TU, Vermaseren, Vogt '17;
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Toolkit for symbolic manipulation of very big expressions

https://www.nikhef.nl/~form/
FORM 4.2: Ruijl, TU, Vermaseren '17
FORM 5.0: coming soon

For physicists & mathematicians:

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manipulations (e.g., substitutions) on each term in your expressions

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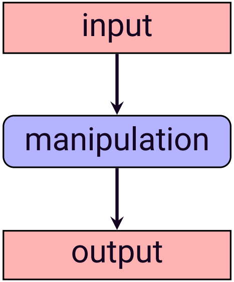

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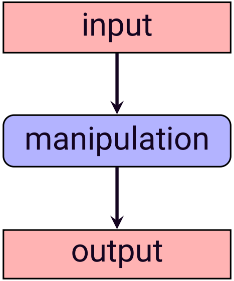
For computer scientists:



may not fit in memory

expressions may blow up

may not fit in memory

Problem:
parallel streaming sort (disk to disk),
but the term size unfixed,
the number of terms varies,
sometimes terms get merged or
cancel out

$$xy + 2xy \to 3xy$$

$$xy - xy \to 0$$

FORM is a physicist's solution for this problem

Integration-by-parts identities (IBPs)

Chetyrkin, Tkachov '81

The divergence theorem in the D-dimensional space-time

$$\int d^D k \frac{\partial}{\partial k^\mu} f^\mu = 0$$

gives linear relations among Feynman integrals

Example: 1-loop massless scalar self-energy

$$F(n_1, n_2) = \int d^D p \frac{1}{(p^2)^{n_1}[(Q-p)^2]^{n_2}}$$

$Q^2 = 1$

$$(D - 2n_1 - n_2)F(n_1, n_2) - n_2 F(n_1 - 1, n_2 + 1) + n_2 F(n_1, n_2 + 1) = 0$$

$$(D - n_1 - 2n_2)F(n_1, n_2) - n_1 F(n_1 + 1, n_2 - 1) + n_1 F(n_1 + 1, n_2) = 0$$

Not all Feynman integrals are independent; reduction to master integrals (MIs)

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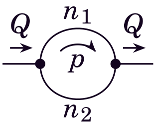

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Not all Feynman integrals are independent; reduction to master integrals (MIs)

Integration-by-parts identities (IBPs)

Chetyrkin, Tkachov '81

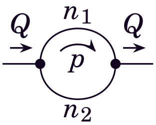
The divergence theorem in the D-dimensional space-time

$$\int d^D k \frac{\partial}{\partial k^\mu} f^\mu = 0$$

gives linear relations among Feynman integrals

Example: 1-loop massless scalar self-energy

$$F(n_1, n_2) = \int d^D p \frac{1}{(p^2)^{n_1}[(Q-p)^2]^{n_2}}$$



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Forcer approach

Extension of the 3-loop Mincer approach to 4-loops

Applies reduction rules and/or symmetries determined from the graph structure (topology) to obtain simpler integrals

Chetyrkin, Kataev, Tkachov '80;
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Triangle rule from IBPs: Chetyrkin, Tkachov '81;
Diamond extension: Ruijl, TU, Vermaseren '15

Automatised code generation

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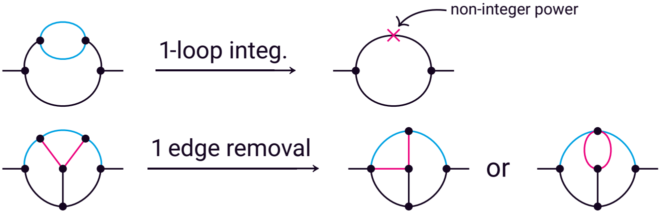

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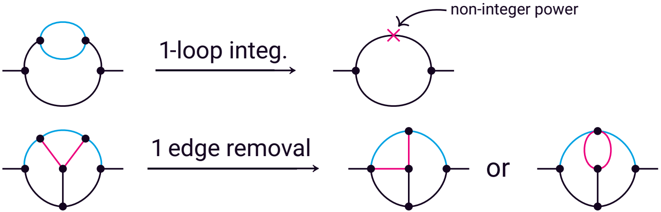

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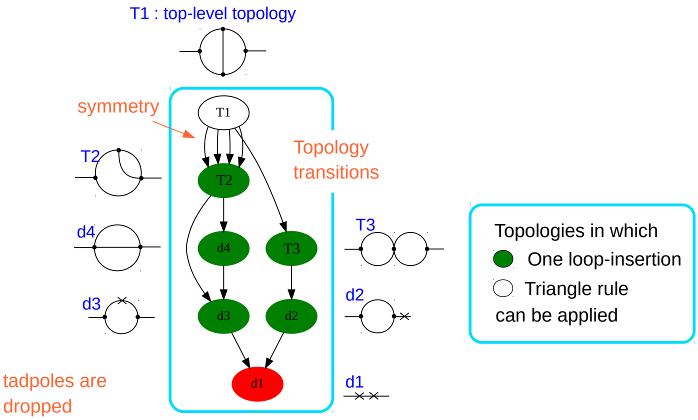
Automatised code generation

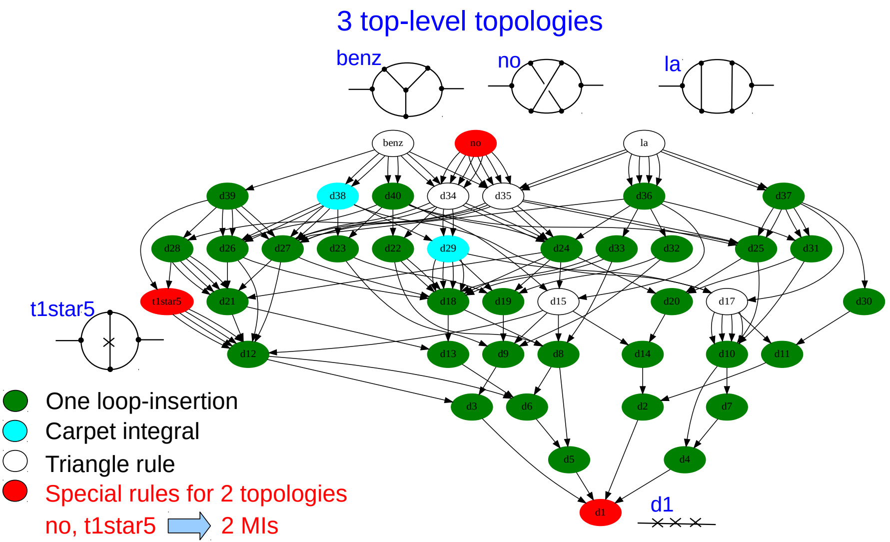
2-loop reduction flowchart

Use the triangle rule and perform one-loop integrals



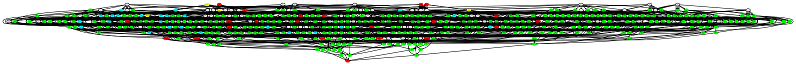
All integrals are expressed in terms of gamma functions

3-loop reduction flowchart (≃Mincer)



4-loop reduction flowchart (Forcer)

All possible topologies classified



416 out of 437 topologies are straightforward to simplify thanks to their structure (1-loop integration/triangle rule etc.)

The remaining 21 topologies require special rules
Constructed manually from IBPs, but considerably computer-assisted, heuristics/brute-force search, sometimes optimised on a trial and error basis

Hand-coding is impossible—write a program to automatically generate the flowchart and reduction code—a 2,794-line Python script (using `igraph`) generated 39,406 lines of FORM code

4-loop reduction flowchart (Forcer)

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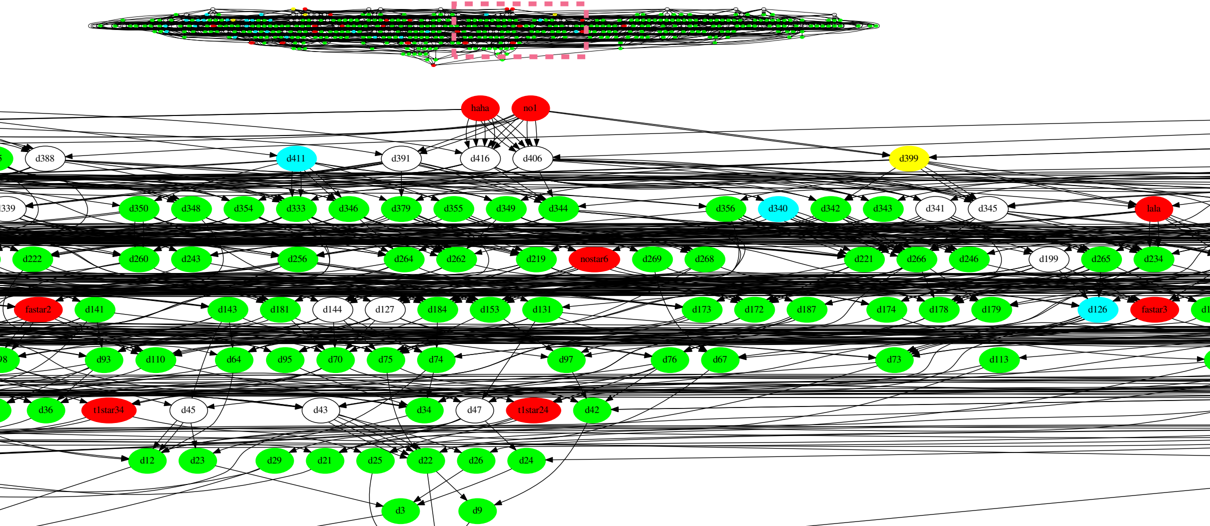

4-loop reduction flowchart (Forcer)

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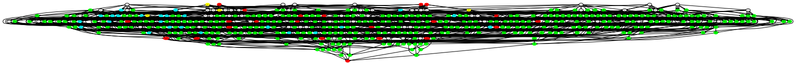

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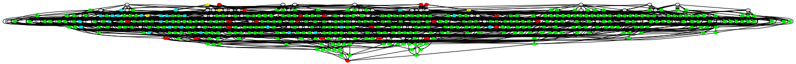

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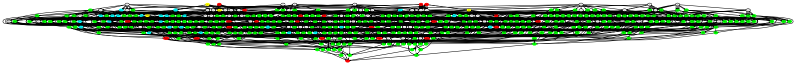

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May the Forcer be with you

Forcer is a FORM program for 4-loop massless propagator-type scalar Feynman integrals

Reduction to the master integrals (MIs) / optionally Laurent expansion in ϵ

MIs all known: Baikov, Chetyrkin '10; Lee, Smirnov, Smirnov '11

Example: "no1" topology

$Q^2 \neq 0$

$$\int d^D p_1 d^D p_2 d^D p_3 d^D p_4 \frac{(2p_2 \cdot p_4)^{-n_{12}} (2Q \cdot p_2)^{-n_{13}} (2Q \cdot p_3)^{-n_{14}}}{(p_1^2)^{n_1} \dots (p_{11}^2)^{n_{11}}}$$

($n_1, \dots, n_{11}$: integers, $n_{12}, \dots, n_{14}$: non-positive integers)

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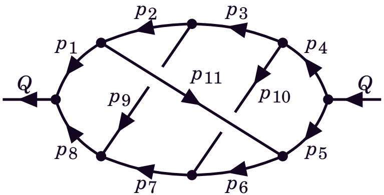

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Benchmark: `no1(2,2,2,2,2,2,2,2,2,2,2,-1,-1,-1)`,
complexity 14 relative to the MI `no1(1,1,1,1,1,1,1,1,1,1,1,0,0,0)`

```
[no1(2,2,2,2,2,2,2,2,2,2,2,-1,-1,-1)] =

-10/9*num(1+2*ep)^2*num(2+5*ep)*num(3+2*ep)^2*num(3+5*ep)*num(4+5*ep)*num(6+5*ep)*num(7+5*ep)*num(8+5*ep)*num(9+5
*ep)*num(36141384+167650024*ep+369157793*ep^2+504389598*ep^3+470560515*ep^4+312347786*ep^5+149770838*ep^6+
51734214*ep^7+12600912*ep^8+2060632*ep^9+203648*ep^10+9216*ep^11)*den(1+ep)^2*den(2+ep)^2*den(2+3*ep)*den(3+ep)^2
*den(4+3*ep)*den(5+3*ep)*den(7+3*ep)*den(8+3*ep)*Master(no1)

+ (20 terms)
```

(rational function of ϵ) $\times$ MI

Exact result: 3.7 hours on a desktop PC with 4 threads

Computations

Feynman diagrams for the background propagator up to 5 loops generated using `QGRAF` Nogueira '93
→ >100k 5-loop diagrams

Use FORM to apply Feynman rules, determine topologies, calculate colour factors etc., merge diagrams with the same factor
→ 9414 5-loop meta diagrams

Local R^* operation implemented in FORM interfaced with Forcer

The total CPU time for the 5-loop diagrams was 3.8×10^7s ($\simeq$ 440days)
Parallelised as much as possible with $>$500 cores, finished in 3 days

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Discussion

QCD result

$$\beta_0 = 11 - \frac{2}{3} n_f$$

$$\beta_1 = 102 - \frac{38}{3} n_f$$

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2$$

$$\beta_3 = \frac{149753}{6} + 3564\zeta_3 + n_f\left(-\frac{1078361}{162} - \frac{6508}{27}\zeta_3\right) + n_f^2\left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) + \frac{1093}{729} n_f^3$$

$$\beta_4 = \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 + n_f\left(-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right)$$
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$$+ n_f^4\left(\frac{1205}{2916} - \frac{152}{81}\zeta_3\right)$$

Substituted: $T_F = 1/2$, $C_A = 3$, $C_F = 4/3$, $d_A^{abcd} d_A^{abcd}/N_A = 135/8$
$d_F^{abcd} d_A^{abcd}/N_A = 15/16$, $d_F^{abcd} d_F^{abcd}/N_A = 5/96$

Agrees with result of Baikov, Chetyrkin, Kühn

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Series convergence

$$\widetilde{\beta} := \frac{\beta}{\beta_{\mathsf{LO}}}$$

$$\widetilde{\beta}(\alpha_s, n_f = 3) = 1 + 0.565884\alpha_s + 0.453014\alpha_s^2 + 0.676967\alpha_s^3 + 0.580928\alpha_s^4$$

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$$\widetilde{\beta}(\alpha_s, n_f = 5) = 1 + 0.401347\alpha_s + 0.149427\alpha_s^2 + 0.317223\alpha_s^3 + 0.080921\alpha_s^4$$

$$\widetilde{\beta}(\alpha_s, n_f = 6) = 1 + 0.295573\alpha_s - 0.029401\alpha_s^2 + 0.177980\alpha_s^3 + 0.001555\alpha_s^4$$

No evidence of any increase of the coefficients indicative of a non-convergent perturbative expansion

Convergence enhanced for larger n_f

Series convergence

$$\widetilde{\beta} := \frac{\beta}{\beta_{\mathrm{LO}}}$$

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No evidence of any increase of the coefficients indicative of a non-convergent perturbative expansion

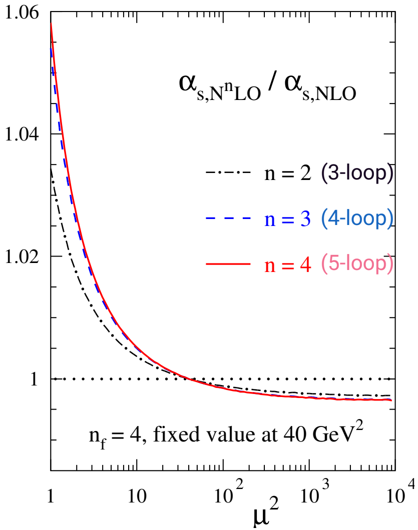
Convergence enhanced for larger n_f

Scale evolution

As an illustrative example,
hypothetically fix $\alpha_s(Q^2 = 40\,\mathrm{GeV}^2) = 0.2$
and assume 4 of active flavours

Even at rather low scales,
the 5-loop contributions are much smaller
than the 4-loop contributions

0.4% vs. 2% at $1\,\mathrm{GeV}^2$



Conclusion

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5-loop beta function in QCD-like theories, valid for a set of n_f-flavour fermions in any representation of any simple Lie group

Computed via local R^* + Forcer within the background field method

Results consistent with other groups

At 5-loop level, the strong coupling evolution now under perfect control

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